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FORWARD UNCERTAINTY PROPAGATION FOR FINITE ELEMENT MODELS WITH NON-GAUSSIAN PARAMETERS

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Abstract. *Due to the inherent uncertainties in manufacturing properties and intrinsic variability of materials, the assumption of homogeneous input variables is generally not justified. As a result, stochastic forward problems have emerged as a tool to incorporate these uncertainties into numerical simulations, improving model prediction capability in structural analyses. Although most of the existing methods focus on the consideration of stochastic loading, the recently developed statFEM employs a Bayesian paradigm to incorporate data and propagate uncertainties from random physical properties in finite element models. This tool, however, is not developed for cases when Gaussian assumptions are inadequate. The present work provides a copula-based approach embedded into the statFEM methodology to propagate uncertainty from arbitrarily distributed physical properties. The random variables are defined in terms of a Gaussian Copula Process, where samples are drawn from a latent variable governed by a Gaussian Process and then brought respectively to copula and marginal spaces, producing random variables with the desired distribution while retaining the usually desired smooth Gaussian dependence in the spatial domain. The quality of the approximated results is then assessed in a simplified 1D Poisson problem by comparing with Monte Carlo sampling results for different random diffusion coefficients, demonstrating that the method is capable of providing good responses for non-Gaussian physical parameters.*

1 INTRODUCTION

The manufacturing of industrial components and the performance of engineering systems usually relies on finite element (FE) models to better determine their physical behavior. These processes are laden with uncertainties, as material properties, geometry, and loading may vary over time and under repeated production. For that end, many physical quantities can be defined by a Gaussian random field (or transformations thereof), ensuring a smooth spatial dependence that often reflects the natural continuity in materials [1].

Alternatively, copula-based approaches can also be used to determine a dependence structure between multiple random variables. Despite being more commonly used in fields such as risk analysis and financial assessment to model time variability, they have also been employed for characterizing stochasticity in the spatial domain [2]. As a feature, copula functions decouple the dependence between random variables from their marginal distributions; but, in the trade-off, the variations within the spatial domain usually do not present a smooth behavior.

In order to propagate the uncertainties to the analysis output, e.g. displacements, statFEM is a statistical tool that provides a methodology for synthesizing measurement data and finite element models [3]. It decomposes the observed data and employs a Bayesian approach to update prior beliefs. In turn, the prior for the finite element component is obtained through a probabilistic forward problem by adopting Gaussian parameters and solving a linear stochastic partial differential equation. It has been used for real-world applications [4], and has also been enhanced with other approaches to solve nonlinear PDEs [5], but solely considering stochastic forcing.

The stochastic behavior in the physical properties is usually not accounted for within stochastic forward problem frameworks, as it complicates the uncertainty propagation process; and even when it is, e.g., as a random diffusion coefficient in [3], it is limited to a Gaussian-distributed variable. However, the quantities related to material properties and geometry are often bounded by conventional standards. Moreover, if there is a lack of information on a random variable, not assuming a bell-shaped distribution may be reasonable. In both cases, using Gaussian marginal distributions is not suitable. To address this gap, a copula-based statFEM approach is proposed to propagate the uncertainty from random physical properties with arbitrary marginal distributions, while keeping the desired smooth spatial variability on the domain. The effectiveness of the presented methodology is assessed by a 1D Poisson toy problem with distinct input distributions.

The rest of the paper is outlined as follows: in Section 2, the statFEM framework is explained, focusing on the stochastic forward problem. In Section 3, the concepts of copulas and copula processes are both presented, while in 3.3, the proposed novel methodology is detailed. The method is applied to a case study in Section 4, presenting results and discussion. Finally, the conclusions of the paper are drawn in Section 5.

2 STATISTICAL FINITE ELEMENT METHOD (statFEM)

This section briefly expands on the statFEM original formulation, which employs a Bayesian statistical approach to integrate finite element models and measurement data. It additively decomposes data into three components: a scaled FE prior, a model misspecification error, and a noise component. Although Bayes' rule can then be used to infer posterior densities and hyperparameters, the main contribution in this study is related to the FE prior response; hence, solely this part of the formulation is briefly introduced. The interested reader is referred to [3] for an in-depth description.

2.1 Probabilistic forward formulation

For the FE stochastic forward problem, the Poisson equation is considered on a domain $\Omega \subset \mathbb{R}^n$ and boundary $\partial\Omega$:

$$\begin{aligned} -\nabla \cdot (\mu(\mathbf{x})\nabla u(\mathbf{x})) &= f(\mathbf{x}), \quad \text{in } \Omega \\ u(x) &= 0, \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $u(\mathbf{x}) \in \mathbb{R}$ is the unknown field, ∇ is the vector differential operator, $\mu(\mathbf{x})$ and $f(\mathbf{x})$ are respectively a random diffusion coefficient and a random forcing term. Both diffusion coefficient and forcing fields are assumed to be Gaussian Processes:

$$\begin{aligned} \mu(\mathbf{x}) &\sim \mathcal{GP}(\overline{\mu}(\mathbf{x}),\, c_\mu(\mathbf{x}, \mathbf{x}') \\ f(\mathbf{x}) &\sim \mathcal{GP}(\overline{f}(\mathbf{x}),\, c_f(\mathbf{x}, \mathbf{x}') \end{aligned} \tag{2}$$

where $\overline{\mu}(\mathbf{x}) = \mathbb{E}[\mu(\mathbf{x})]$ and $\overline{f}(\mathbf{x}) = \mathbb{E}[f(\mathbf{x})]$ are simplified to be mean values evaluated point-wise, c_μ and c_f are the covariance kernels. A squared exponential kernel is adopted in both cases:

$$c_\alpha(\mathbf{x}, \mathbf{x}') = \sigma_\alpha^2 \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|}{2l_\alpha^2}\right) \tag{3}$$

in which $\sigma_\alpha \in \mathbb{R}^+$ is the scaling parameter and $l_\alpha \in \mathbb{R}^+$ is the length scale parameter of the random variable α.

After the domain Ω is subdivided into a set of non-overlapping elements, and the unknown field $u(\mathbf{x})$ is approximated by linear Lagrange basis functions, the discretization of the weak form in Eq. (1) yields the linear system of equations:

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{u} = \mathbf{f} \tag{4}$$

where $\mathbf{A}(\boldsymbol{\mu})$ is the system matrix, $\boldsymbol{\mu}$ is the vector of diffusion coefficients, $\mathbf{u}$ is the nodal displacements vector, and $\mathbf{f}$ is the nodal source vector.

The diffusion coefficient is assumed constant in each element, being evaluated at the element centroids. The diffusion coefficient vector can then be expressed by the multivariate Gaussian density:

$$\boldsymbol{\mu} \sim p(\boldsymbol{\mu}) = \mathcal{N}\left(\overline{\boldsymbol{\mu}}\left(\mathbf{X}^{(c)}\right),\, \boldsymbol{\Sigma}_\mu\left(\mathbf{X}^{(c)}, \mathbf{X}^{(c)}\right)\right) \tag{5}$$

where the mean vector $\overline{\boldsymbol{\mu}}$ and covariance matrix $\boldsymbol{\Sigma}_\mu$ are evaluated with respect to the matrix of centroid coordinates $\mathbf{X}^{(c)}$. Then, the element system matrices are defined by:

$$\mathbf{A}_{ij}^e = \int_{\Omega_e} \mu_e \frac{\partial \phi_i(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial \phi_j(\mathbf{x})}{\partial \mathbf{x}}\, d\Omega_e \tag{6}$$

in which μ_e is the diffusion coefficient of the element with index e and $\phi_{i,j}$ are from the local set of shape functions. On the other hand, the source vector is given by the multivariate Gaussian density:

$$\mathbf{f} \sim p(\mathbf{f}) = \mathcal{N}(\bar{\mathbf{f}}, \boldsymbol{\Sigma}_f). \tag{7}$$

Next, it is possible to express the probability density for a solution vector $\mathbf{u}$ in terms of a diffusion coefficient $\boldsymbol{\mu}$ by solving Eq. (4):

$$\mathbf{u} = \mathbf{A}(\boldsymbol{\mu})^{-1}\mathbf{f} \tag{8}$$

If $\boldsymbol{\mu}$ is known, the right-hand side of the equation is a linear transformation of the Gaussian source vector $\mathbf{f}$ such that:

$$p(\mathbf{u}|\boldsymbol{\mu}) = \mathcal{N}\left(\mathbf{A}(\boldsymbol{\mu})^{-1}\bar{\mathbf{f}}, \mathbf{A}(\boldsymbol{\mu})^{-1}\boldsymbol{\Sigma}_f\mathbf{A}(\boldsymbol{\mu})^{-\intercal}\right) \tag{9}$$

Consequently, the unconditional density can be obtained by marginalizing it over $\boldsymbol{\mu}$, yielding:

$$p(\mathbf{u}) = \int p(\mathbf{u}|\boldsymbol{\mu})p(\boldsymbol{\mu})\, d\boldsymbol{\mu} \tag{10}$$

This integral can be evaluated numerically using either Monte Carlo or Markov Chain Monte Carlo sampling, for example. However, since this approach may be impractical for large scale problems, a first-order approximation for $p(\mathbf{u})$ is given by the multivariate Gaussian density:

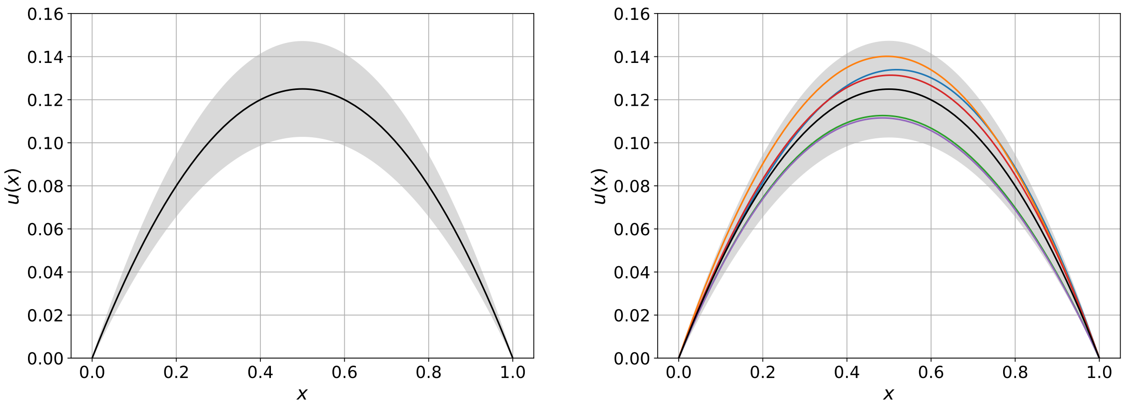
$$p(\mathbf{u}) = \mathcal{N}(\bar{\mathbf{u}}, \boldsymbol{\Sigma}_u) \tag{11}$$

where:

$$\begin{aligned}
\bar{\mathbf{u}} &= \mathbf{A}(\overline{\boldsymbol{\mu}})^{-1}\bar{\mathbf{f}} \\
\boldsymbol{\Sigma}_u &= \mathbf{A}(\overline{\boldsymbol{\mu}})^{-1}\boldsymbol{\Sigma}_f\, \mathbf{A}(\overline{\boldsymbol{\mu}})^{-\intercal} + \\
&\sum_e \sum_d (\boldsymbol{\Sigma}_\mu)_{ed}\, \mathbf{A}(\overline{\boldsymbol{\mu}})^{-1} \frac{\partial \mathbf{A}(\overline{\boldsymbol{\mu}})}{\partial \mu_e} \mathbf{A}(\overline{\boldsymbol{\mu}})^{-1} (\boldsymbol{\Sigma}_f + \bar{\boldsymbol{f}} \otimes \bar{\boldsymbol{f}})\, \mathbf{A}(\overline{\boldsymbol{\mu}})^{-\intercal} \frac{\partial \mathbf{A}(\overline{\boldsymbol{\mu}})^{\intercal}}{\partial \mu_d} \mathbf{A}(\overline{\boldsymbol{\mu}})^{-\intercal}
\end{aligned} \tag{12}$$

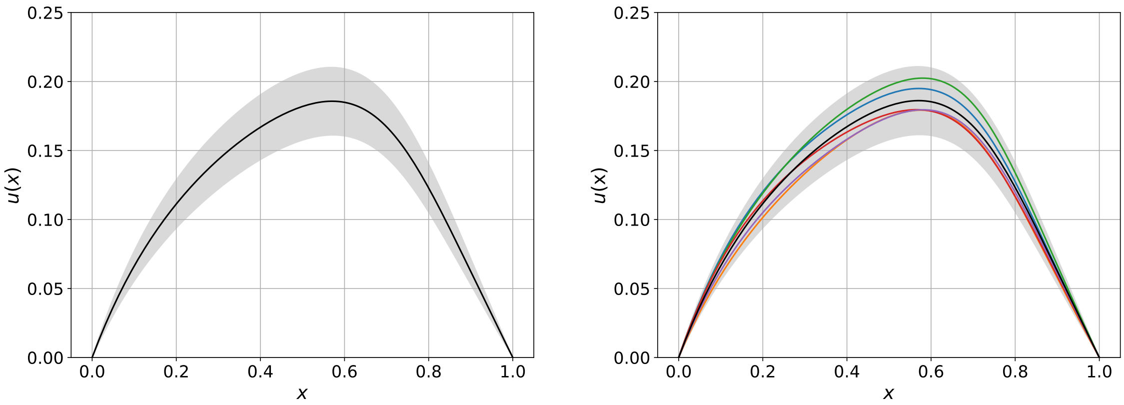
in which the indices e, d are referring to arbitrary elements, as the process is made element-wise.

The goodness of this approximation is assessed by comparing it against the much more computationally expensive sampling output. As an example, in Fig. 1 the forcing term is stochastic, with parameters $\overline{f}(x) = 1$, $\sigma_f = 0.1$, $l_f = 0.4$, and deterministic diffusion coefficient $\mu = 1$; whereas in Fig. 2 the diffusion coefficient is stochastic, with parameters $\overline{\mu}(x) = 0.7 + 0.3\sin(2\pi x)$, $\sigma_\mu = 0.1$, $l_\mu = 0.25$, and deterministic forcing $f = 1$. It is possible to observe that the approximation provides a good agreement with the numerical response in both cases. The colored lines are samples drawn from Eq. (9) for Fig. 1 and from Eq. (5) then $u(x, \mu)$ for Fig. 2, while the grey shaded areas are the 95% confidence regions for each case.



(a) StatFEM solution. (b) MC sampling solution.

Figure 1: Comparison between (a) StatFEM and (b) MC sampling for stochastic f. Grey shaded areas represent the 95% confidence regions.



(a) StatFEM solution. (b) MC sampling solution.

Figure 2: Comparison between (a) StatFEM and (b) MC sampling for stochastic μ. Grey shaded areas represent the 95% confidence regions.

3 COPULA-BASED STATFEM

Throughout this section, a novel methodology for considering stochasticity in physical properties is proposed, with a copula approach being embedded into the previously defined statFEM formulation for uncertainty propagation. The concepts of copulas and copula processes are presented in Sections 3.1 and 3.2, respectively. Then, in Section 3.3, the methodology is explained for both the numerical (sampling) and the linear approximation solutions.

3.1 Copulas

The configuration of scatter plots between random variables is often influenced by their scales and marginal distributions. To better assess their dependence structure, the probability integral transformation can be used in each as a standardization technique, which enables the characterization between uniformly [0,1] distributed random variables. This behavior is illustrated in Fig. (3). Consequently, this approach decouples the dependence structure between random variables from their marginal distributions. More specifically, in this context, the corresponding joint distribution function employed to define the dependence structure between random variables with uniform marginals is known as a copula [6].

Delving into the mathematical foundation, copulas are based on Sklar's Theorem [7]. This theorem states that any multivariate joint distribution can be expressed in terms of both univariate marginal distributions and a copula that captures the dependence structure between the random variables:

$$H(y_1, y_2, ..., y_n) = C(F_1(y_1), F_2(y_2), ..., F_n(y_n)) = C(z_1, z_2, ..., z_n) \tag{13}$$

where H represents the joint distribution function, C is the copula, and F_i denotes the cumulative distribution function (CDF) of each random variable. It is important to remark that, if all F_i are continuous, the copula C is unique.

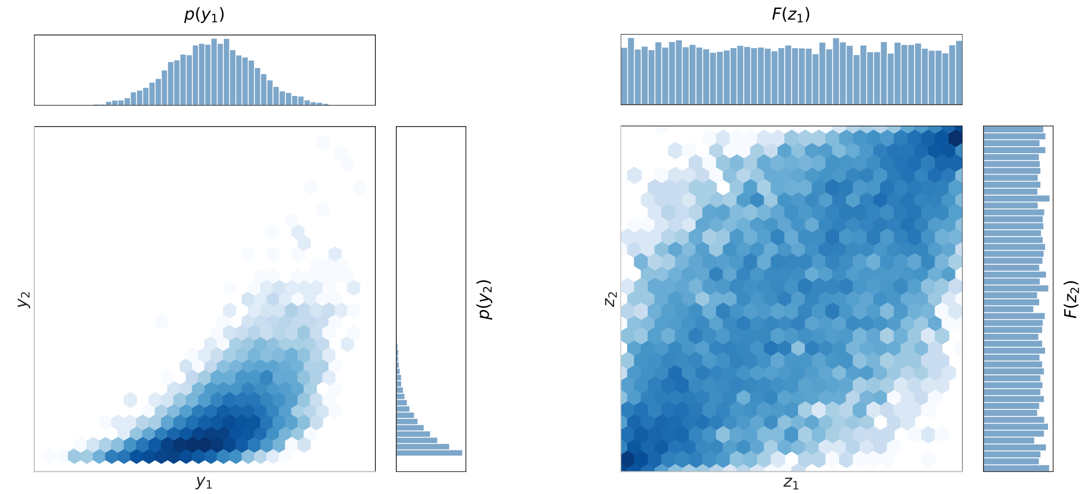
Furthermore, the inverse relation also holds, allowing the copula to be defined as:

$$C(z_1, z_2, ..., z_n) = H(F_1^{-1}(z_1), F_2^{-1}(z_2), ..., F_n^{-1}(z_n)) \tag{14}$$

This formulation can then be used to construct various types of copulas [8]. This work, however, focuses on the Gaussian copula due to its simplicity and computational efficiency. The multivariate Gaussian copula is given by:

$$C(\mathbf{u}; \Sigma) = \Phi_\Sigma(\Phi^{-1}(z_1), \Phi^{-1}(z_2), ..., \Phi^{-1}(z_n)) \tag{15}$$

where Φ^{-1} denotes the inverse standard normal CDF, and Φ_Σ represents the standard multivariate normal distribution with a zero mean vector, unit variances, and symmetric positive definite correlation matrix Σ. Consequently, by substituting $z_i = F_i(y_i)$, the joint distribution H is defined by a Gaussian dependence and arbitrary F_i marginals, as long as they are continuous.



(a) Scatter plot based on probability distribution functions.

(b) Scatter plot based on cumulative distribution functions.

Figure 3: Schematic example of a copula function.

3.2 Copula Processes

Consider a latent variable $\kappa(\mathbf{x})$ that follows a Gaussian process. Sampling from this process yields a collection of Gaussian variables, with their spatial dependence structure encoded by a chosen kernel function. More specifically, in this work, the latent variable $\kappa(\mathbf{x})$ is such that:

$$\kappa(\mathbf{x}) \sim \mathcal{GP}(\overline{\kappa}(\mathbf{x}),\ c_\kappa(\mathbf{x}, \mathbf{x}')) \tag{16}$$

where:

$$\overline{\kappa}(\mathbf{x}) = 0, \quad c_\kappa(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|}{2l_\kappa^2}\right) \tag{17}$$

Thus, the dependence structure is solely influenced by the length scale l_κ. These sampled values are then transformed using a Gaussian CDF, providing a collection of $\mathbf{Z}$ uniformly [0,1] distributed latent random variables:

$$\mathbf{Z} = \Phi(\kappa) \tag{18}$$

Consequently, although the samples transition from the latent space to the copula space, their dependence structure remains governed by the underlying Gaussian process. Thus, these resulting uniform samples can be defined as a draw from a Gaussian Copula Process, due to both the inherited dependence structure and the type of CDF used [9].

Moreover, these samples in copula space can also be mapped to another arbitrary marginal distribution by applying the desired CDF, transitioning from copula space to marginal space:

$$\underbrace{\kappa(x) \sim \mathcal{GP}\big(0, c_\kappa(\mathbf{x}, \mathbf{x}')\big)}_{\text{Step 1: Sample from latent GP}} \xrightarrow{\text{Gaussian CDF}} \underbrace{\mathbf{Z} = \Phi(\kappa)}_{\text{Step 2: Map to copula space}} \xrightarrow{\text{Inverse CDF}} \underbrace{\mathbf{Y} = \mathbf{F}^{-1}(\mathbf{Z})}_{\text{Step 3: Transform to marginals}} \tag{19}$$

For instance, if a beta inverse CDF is used, the transformed samples will follow a beta distribution while still maintaining the underlying Gaussian dependence structure. Therefore, this methodology provides an efficient approach to generate samples from any desired marginal distribution while promoting a smooth variation within the spatial domain.

3.3 Incorporating copulas in statFEM

The question that remains is how to arrive at a formulation that provides a version of statFEM that incorporates the Copula Process approach. Primarily, it is necessary to determine how it affects the evaluation of the mean approximation. Given that the diffusion coefficient follows an arbitrary marginal distribution, it is possible to evaluate the expected value over the domain $\overline{\mu}(\mathbf{x}) = \mathbb{E}[\mu(\mathbf{x})]$. Then, this value can be substituted in Eq. (12) to assemble the system matrix and subsequently evaluate the approximated mean.

Next, for the evaluation of the approximated covariance, $\mathbf{\Sigma}_\mu$ needs to be determined. Since a perturbation method is used to evaluate the first order approximation in statFEM [10], the same approach is employed to find the relation between $\mathbf{\Sigma}_\mu$ and $\mathbf{\Sigma}_\kappa$. Given that there is a mapping such that $\mu(x) = g(\kappa(x))$, a first-order Taylor expansion around $\overline{\kappa}$ (with zero mean in this case) results in:

$$g(\kappa(\mathbf{x})) \approx g(0) + g'(0)\kappa(\mathbf{x}) \tag{20}$$

where $g(0)$ is the expected value of μ when $\kappa(x) = 0$, and $g'(0)$ addresses how a small perturbation in $\kappa(\mathbf{x})$ leads to a change in μ. The perturbation of μ is:

$$\delta\mu = \mu - g(0) \xrightarrow{\text{Linearization}} \delta\mu \approx g'(0)\kappa(\mathbf{x}) \tag{21}$$

Then, the covariance of $\delta\mu$ is given by:

$$\mathbf{\Sigma}_\mu = \text{Cov}(\delta\mu, \delta\mu) \approx \left(g'(0)\right)^2 \text{Cov}(\kappa(\mathbf{x}), \kappa(\mathbf{x})) = \left(g'(0)\right)^2 \mathbf{\Sigma}_\kappa \tag{22}$$

Thus, the propagated uncertainty in μ can be defined in terms of the latent variable κ and the sensitivity $g'(0)$, which depends on the chosen marginal distribution function. Finally, $\mathbf{\Sigma}_\mu$ can be substituted into Eq. (12), along with the previously obtained system matrix, and be used to evaluate the output covariance $\mathbf{\Sigma}_u$.

Furthermore, once the response is obtained, the goodness of the approximation can be assessed by comparing it with MC sampling from the probability $p(\mathbf{u})$ defined in Eq. (10). In each run, a single realization from $\kappa(\mathbf{x})$ in Eq. (16) is drawn, and the procedure defined in Section 3.2 is employed to obtain samples from the desired marginal space, which are then used to assemble the system matrix.

4 APPLICATION CASE

In this section, the proposed Copula-based statFEM is applied to a one-dimensional Poisson problem, where the domain is discretized into 32 elements using linear Lagrange shape functions. In Section 4.1, the governing equations and stochastic variables are defined, and in Section 4.2 the employment of the copula process to the problem at hand is discussed. The results are presented in Section 4.3 and discussed in Section 4.4.

4.1 Case definition

The one-dimensional Poisson-Dirichlet problem is defined based on Eq. (1):

$$\begin{aligned} -\frac{d}{dx}\left(\mu(x)\frac{du}{dx}\right) &= f(x), \quad \text{in } \Omega = (0,1) \\ u(x) &= 0, \qquad \text{on } x=0 \text{ and } x=1 \end{aligned} \tag{23}$$

where the diffusion coefficient $\mu(x)$ is random, and the forcing term $f(x) = 1$ is deterministic. The solution $u(x)$ is evaluated using the copula-based statFEM approach, while the quality of the approximations is assessed by comparing with MC sampling results and employing the Wasserstein 2-distance (W_2) as a dissimilarity metric. This is particularly advantageous when dealing with MC samples as it can be directly applied to point clouds, not requiring the empirical distribution to be functionally described [1].

4.2 Probabilistic FE analysis

Since the proposed approach is supposed to deal with arbitrary marginal distributions, two different ones are used to represent the random diffusion coefficient. In both cases, however, μ is assumed to have a known expected value ($\overline{\mu} = \mathbb{E}[\mu(x)] = 1$) and bounds ($\pm\, 0.2\,\overline{\mu}$). For a diffusion coefficient that follows a uniform distribution:

$$\mu(x) \sim \mathcal{U}(0.8, 1.2) \implies g(\kappa) = \mu_{min} + (\mu_{max} - \mu_{\min})\,\Phi(\kappa) \tag{24}$$

The evaluation of the approximated mean and covariance from Eq. (12) requires the definition of:

$$\begin{aligned} \overline{\mu} &= \mathbb{E}[\mu(x)] = 1 \\ \mathbf{\Sigma}_\mu &\approx \left(\frac{\mu_{max} - \mu_{min}}{\sqrt{2\pi}}\right)^2 \mathbf{\Sigma}_\kappa \end{aligned} \tag{25}$$

where both μ_{min}, μ_{max} refer to the bounds. Alternatively, for the case it follows a beta distribution:

$$\begin{aligned} &X \sim \text{Beta}(5,5) \\ &\mu(x) = 0.8 + (1.2 - 0.8)X \implies g(\kappa) = \mu_{min} + (\mu_{max} - \mu_{\min})F^{-1}_{Beta}(\Phi(\kappa); 5, 5) \end{aligned} \tag{26}$$

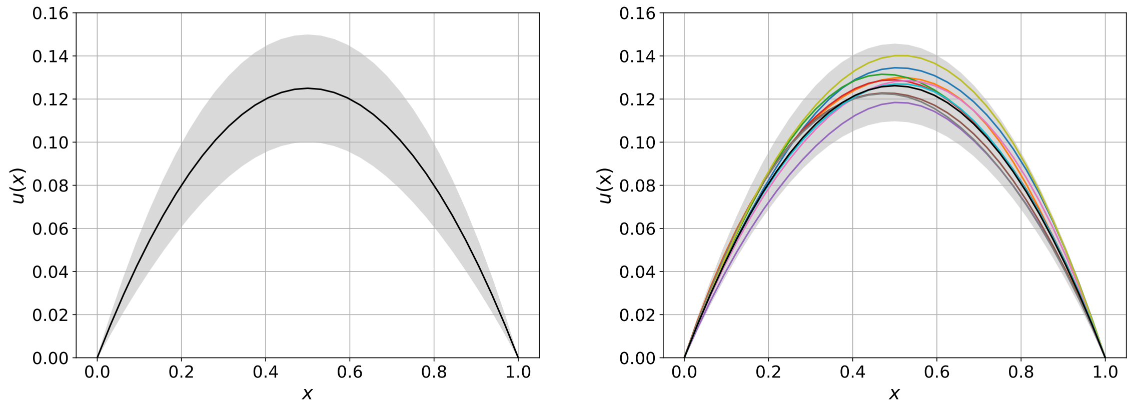
where $F^{-1}_{Beta}(\cdot; \alpha, \beta)$ is the inverse beta CDF. Thus:

$$\begin{aligned} \overline{\mu} &= \mathbb{E}[\mu(x)] = 1 \\ \mathbf{\Sigma}_\mu &\approx \left(\frac{\mu_{max} - \mu_{min}}{\sqrt{2\pi}} \frac{1}{f_{Beta}\left(F^{-1}_{Beta}(0.5; 5, 5)\right)}\right)^2 \mathbf{\Sigma}_\kappa \end{aligned} \tag{27}$$

in which f_{Beta} is the beta probability distribution function. Finally, to completely define the stochastic variables in the proposed model, the kernel function of the latent variable $\kappa(x)$ can be modeled solely in terms of the length scale, as previously defined. In both cases, $\mathbf{\Sigma}_\kappa$ is computed for $l_\kappa = 0.2$.

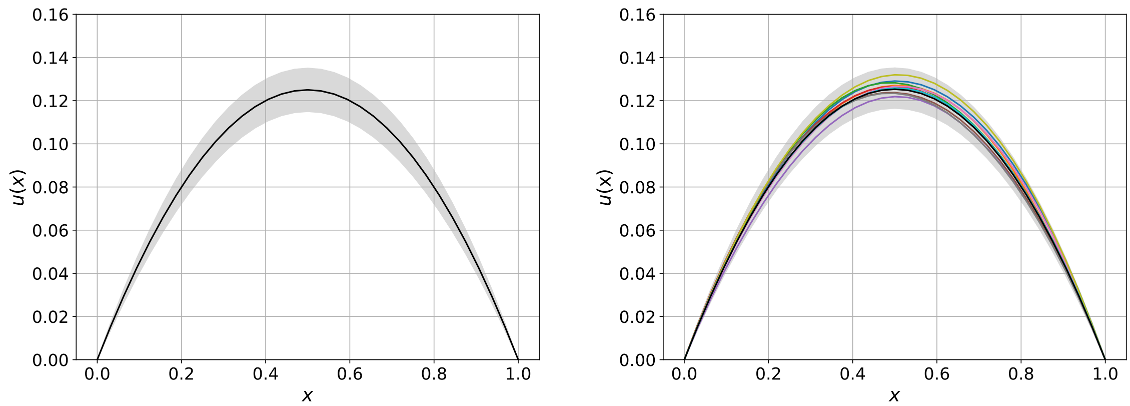
4.3 Results

Since it is now possible to evaluate second-order statistics, i.e., mean and covariance, of the approximated solution field $p(\mathbf{u}) = \mathcal{N}(\overline{\mathbf{u}}, \boldsymbol{\Sigma}_u)$, and also to sample from the input distribution for the MC solution, the methodology described in Section 3.3 can be employed. The displacements for the uniform and beta distributed diffusion coefficients are illustrated, respectively, in Figs. 4 and 5. Then, the moment-matched distribution is sampled and the Wasserstein 2-distances are shown in Table 1 to indicate the dissimilarity between the results and the MC samples in both cases. The colored lines are realizations of $u(x, \mu)$ from samples drawn from Eq. (5), and the shaded areas are the 95% confidence regions for each case.



(a) StatFEM solution. (b) MC sampling solution.

Figure 4: Comparison between (a) StatFEM and (b) MC sampling for uniformly distributed μ. The shaded areas depict the 95% confidence regions.



(a) StatFEM solution. (b) MC sampling solution.

Figure 5: Comparison between (a) StatFEM and (b) MC sampling for beta distributed μ. The shaded areas depict the 95% confidence regions.

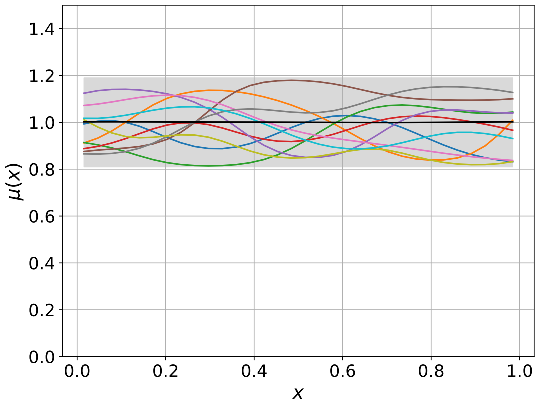
Furthermore, the diffusion coefficient profiles along the domain for the highlighted samples are shown in Fig. 6. The shaded areas are related to the aforementioned confidence interval.

4.4 Discussion

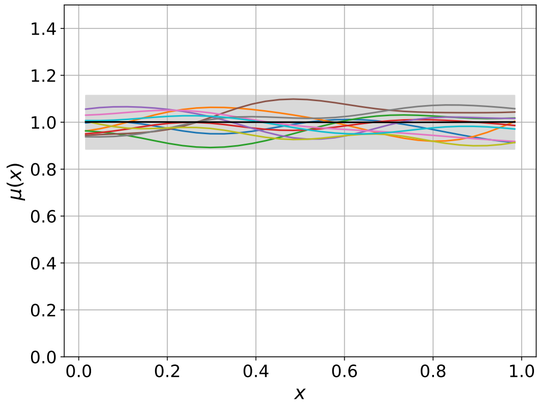
Observing both Figs. 4 and 5, the mean, due to the same expected values in both distributions, is in excellent agreement with reference results. However, the uniformly distributed

$\mu(x)$	W_2
Uniform	0.069
Beta	0.031

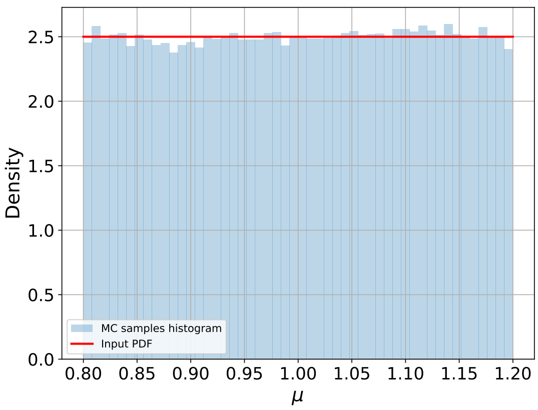
Table 1: Wasserstein 2-distances for different input distributions.



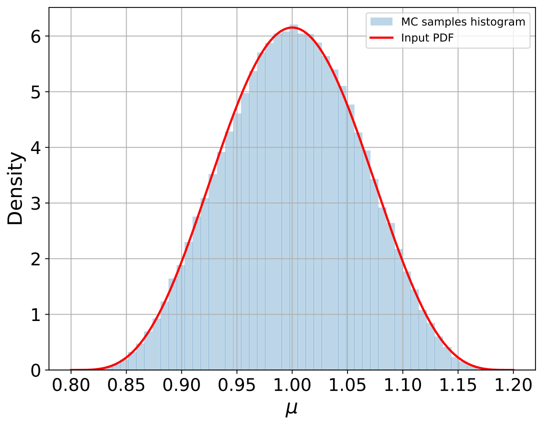
(a) Uniformly-distributed μ profiles.



(b) Beta-distributed μ profiles.



(c) Uniformly-distributed μ histogram.



(d) Beta-distributed μ histogram.

Figure 6: Characterization of μ samples with different marginal distributions.

diffusion coefficient propagates a higher uncertainty to the output than the beta distributed one. This behavior is expected, since the uniform probability density is higher for values farther from the mean than the beta one, resulting in more scattered realizations of μ, as seen in Fig. 6.

Moreover, the approximation in both cases results in higher uncertainty than the reference ($\mathbf{\Sigma_u} > \mathbf{\Sigma}_{\text{MC}}$). This difference is noticeably higher in the uniformly distributed case, while presenting a better agreement for the beta one. The goodness of these approximations is measured in Table 1, which shows that the Beta's W_2 is much lower. This is expected, since the beta distribution is symmetric and bell-shaped, resembling well the assumed Gaussian behavior in the output. However, it is valid to remark that the provided linear approximation for the Uniform marginal can still be used, since the reasonability of the W_2 distance absolute value is context-dependent.

5 CONCLUSIONS

In this work, a Gaussian Copula Process approach is used to generate random physical parameters with arbitrary marginal distributions and a latent, usually desired, Gaussian spatial dependence. It is embedded into the original statFEM framework to have the uncertainty propagated, being seamlessly integrated into numerical simulations. This statistical approach is assessed by solving a probabilistic forward problem for beta and uniformly distributed diffusion coefficients of a 1D Poisson toy problem, with the obtained approximations being compared to MC sampling solutions.

Although the developed copula-based statFEM approach allows for the use of arbitrary marginal distributions, the quality of the resulting Gaussian approximation of the non-Gaussian statFEM marginal over the FE solution is observed to be dependent on the input distribution. Due to the higher resemblance with a Gaussian behavior, the results for the beta distributed μ present a much better agreement with the MC sampling solution. On the other hand, although the approximation for the Uniform marginal case can still be considered reasonable, it is of substantially lower quality. Furthermore, it is important to remark that the majority of this difference is due to the approximated covariance, which was, in both cases, greater than the one obtained from MC sampling. This would, however, be desirable e.g., in the context of structural design, acting as a safety factor regarding the propagated uncertainty while still providing close approximations.

Therefore, this work highlights the suitability of the proposed copula-based statFEM framework to solve probabilistic forward FE problems for non-Gaussian distributed diffusion coefficients, as the presented results demonstrate a good agreement with reference values for different input distributions. The goodness of the linear approximations, however, was shown to be dependent on the marginal distribution of μ. Thus, an improvement on the approximation method to further enhance the framework seems feasible. Additionally, since this work focuses only on the forward problem with smoothly varying random variables, the demonstration of the copula-based statFEM in inverse problems is left for future studies. Other possible extensions are the GP characterization by a Gaussian Copula-based Bayesian Network and the consideration of variables with non-Gaussian dependence.

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